

ON RELATIONS FOR ZEROS OF f -POLYNOMIALS AND f^+ -POLYNOMIALS.

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ABSTRACT. Let Φ be an irreducible (possibly noncrystallographic) root system of rank l of type P . For the corresponding cluster complex $\Delta(P)$, which is known as pure $(l-1)$ -dimensional simplicial complex, we define the generating function of the number of faces of $\Delta(P)$ with dimension $i-1$, which is called the f -polynomial. We show that the f -polynomial has exactly l simple real zeros on the interval $(0, 1)$ and the smallest root for the infinite series of type A_l , B_l and D_l monotone decreasingly converges to zero as the rank l tends to infinity. We also consider the generating function (called the f^+ -polynomial) of the number of faces of the positive part $\Delta_+(P)$ of the complex $\Delta(P)$ with dimension $i-1$, whose zeros are real and simple and are located in the interval $(0, 1]$, including a simple root at $t = 1$. We show that the roots $\{t_{P,\nu+1}^+\}_{\nu=1}^{l-1}$ in decreasing order of f^+ -polynomial alternate with the roots $\{t_{P,\nu}\}_{\nu=1}^l$ in decreasing order of f -polynomial.

1. INTRODUCTION

Let Φ be an irreducible (possibly noncrystallographic) root system of rank l of type $P \in \{A_l \ (l \geq 1), B_l \ (l \geq 2), D_l \ (l \geq 4), E_l \ (l = 6, 7, 8), F_4, G_2, H_3, H_4, I_2(p) \ (p \geq 3)\}$. Let Φ^+ be a positive system for Φ with corresponding simple system Π . The cluster complex $\Delta(P)$ ¹ introduced by Fomin-Zelevinsky ([F-Z]) is a pure $(l-1)$ -dimensional simplicial complex whose ground set is the set $\Phi_{\geq -1} := \Phi^+ \sqcup (-\Pi)$ of almost-positive roots and its geometric realization is homeomorphic to a sphere. Let $\Delta_+(P)$ denote the induced subcomplex of $\Delta(P)$ on the vertex set Φ^+ . The complex $\Delta_+(P)$ is referred to as the *positive part* of $\Delta(P)$. For a non-negative integer i with $0 \leq i \leq d-1$, let f_i (resp. f_i^+) denote the number of faces of $\Delta(P)$ (resp. $\Delta_+(P)$) with dimension $i-1$. We call the sequence $(f_0, f_1, \dots, f_{l-1})$ (resp. $(f_0^+, f_1^+, \dots, f_{l-1}^+)$) the f -vector of the complex $\Delta(P)$ (resp. $\Delta_+(P)$). Then, we define the f -polynomial and f^+ -polynomial of type P in the formal variable t by

$$f_P(t) := \sum_{i=0}^l f_{i-1}(P)(-t)^i, \quad f_P^+(t) := \sum_{i=0}^l f_{i-1}^+(P)(-t)^i,$$

where $f_{-1}(P) := 1$ and $f_{-1}^+(P) := 1$. In this article, we will study the zero loci of the f -polynomial. In a similar manner to [I-S], we will show that the f -polynomial has an unexpected strong connection with orthogonal polynomials. By making the best use of these properties, we will show that the f -polynomial has exactly l simple real zeros on the interval $[0, 1]$ and the smallest root for the infinite series of type A_l , B_l and D_l monotone decreasingly converges to zero as the rank l tends to infinity. Furthermore, we obtain inequalities for zeros of f -polynomials and f^+ -polynomials.

In [I-S], the authors studied the zero loci of the *skew-growth function* ([Sa2])² of a *dual Artin monoid* of finite type ([Be]). They noted that the skew-growth function for the dual

¹In [F-Z], the notion of cluster complex $\Delta(P)$ was originally defined for the case where Φ is crystallographic. As was noted in [F-R2], the constructions in [F-Z] extend verbatim to the noncrystallographic types.

²For a positive homogeneously finitely presented monoid $M = \langle L \mid R \rangle_{mo}$ that satisfies the cancellation condition (i.e. $axb = ayb$ implies $x = y$) and is equipped with a degree map $\deg : M \rightarrow \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words, the *spherical growth function* for the monoid M is defined as

$$P_{M, \deg}(t) := \sum_{u \in M} t^{\deg(u)}.$$

Artin monoid of type P coincides with the f^+ -polynomial (for the complex $\Delta_+(P)$) of the same type³. Suggested by some numerical experiments concerning the f^+ -polynomials, they conjectured the following 1, 2 and 3 ([I-S])⁴.

1. $f_P^+(t)/(1-t)$ is an irreducible polynomial over \mathbb{Z} , up to the trivial factor $1-2t$ for the types A_l (l : even) and D_4 .
2. $f_P^+(t)$ has $l = \text{rank}(P)$ simple real roots on the interval $(0, 1]$, including a simple root at $t = 1$.
3. The smallest root of $f_P^+(t)$ monotonously decreasingly converges to 0 as the rank l tends to infinity for the infinite series of type A_l , B_l and D_l .

By analogy with 1, 2 and 3, and also inspired by some numerical experiments (see Appendix II for the figures of the zero loci of the functions of types A_{20} , B_{20} , D_{20} and E_8), we conjecture the following.

Conjecture 1. $f_P(t)$ is an irreducible polynomial over \mathbb{Z} , up to the trivial factor $1-2t$ for the types A_l (l : odd), B_l (l : odd), D_l (l : odd), E_7 and H_3 .

Conjecture 2. $f_P(t)$ has $l = \text{rank}(P)$ simple real roots on the interval $(0, 1)$.

Conjecture 3. The smallest root of $f_P(t)$ monotonously decreasingly converges to 0 as the rank l tends to infinity for the infinite series of type A_l , B_l and D_l .

Remark 1.1. Conjecture 1. is approved for types A_l ($1 \leq l \leq 30$), B_l ($2 \leq l \leq 30$), D_l ($4 \leq l \leq 30$), E_6 , E_7 , E_8 , F_4 , G_2 , H_3 , H_4 and $I_2(p)$ ($p \geq 3$) by using the software package Mathematica on the Table A and B in Appendix I.

In addition to the above three conjectures, we give one more conjecture on a relation for zeros of f -polynomials and f^+ -polynomials. For $P \in \{A_l$ ($l \geq 1$), B_l ($l \geq 1$), D_l ($l \geq 4$), E_l ($l = 6, 7, 8$), $F_4, G_2, H_3, H_4, I_2(p)$ ($p \geq 3$)}, let $t_{P,\nu}$ be the zeros of $f_P(t)$ in decreasing order (i.e. $1 > t_{P,1} > t_{P,2} > \cdots > t_{P,l} > 0$) and let $t_{P,\nu}^+$ be the zeros of $f_P^+(t)$ in decreasing order (i.e. $1 = t_{P,1}^+ > t_{P,2}^+ > \cdots > t_{P,l}^+ > 0$).

Conjecture 4. The system $\{t_{P,\nu+1}^+\}_{\nu=1}^{l-1}$ alternates with the system $\{t_{P,\nu}\}_{\nu=1}^l$, that is,

$$t_{P,\nu} > t_{P,\nu+1}^+ > t_{P,\nu+1}, \quad (\nu = 1, \dots, l-1).$$

The aim of the present paper is to give affirmative answers to Conjectures 2, 3 and 4. In §2, we first prepare some useful functional equations (2.1), (2.2) and (2.3) for later use. In the functional equation for type A_l (resp. B_l), the f -polynomial of type A_l (resp. B_l) has a simple relation with the f^+ -polynomial of the same type. Although for type D_l the f -polynomial does not have a simple relation with the f^+ -polynomial of the same type, the f -polynomial of type D_l has a rather complicated relation with the f -polynomial of type B_l . Hence, from the equation (2.2) and the equation (6.4) in [I-S], we may say that the f -polynomial of type D_l has a certain relation with f^+ -polynomial of type D_l . Secondly, for the three infinite series A_l , B_l and D_l of f -polynomials, we show that they are expressed in higher (logarithmic) derivatives of the polynomials of the form $t^p(1-t)^q$. In analogy

If any two elements in the monoid M admit left (resp. right) least common multiple, the *skew-growth function* for it is defined as

$$N_{M,\deg}(t) := \sum_{J \subset I_0} (-1)^{\#J} t^{\deg(\text{lcm}_r(J))},$$

where $I_0 := \min_l(M \setminus \{1\})$ (the set of minimal elements of M with respect to the partial ordering induced by the left division relation). Then, the following inversion formula for M holds

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1.$$

This fact is generalized in [Sa2]. The distribution of the zeros of the skew-growth functions are investigated from the viewpoint of limit functions ([Sa3]).

³This fact is proved in [At]Corollary 2.5. In addition to this fact, it is known that the skew-growth function is identified with the generating function of Möbius invariants (called the *characteristic polynomial*) of the *lattice of non-crossing partitions* ([K, Be, B-W]).

⁴In [Sa1], the author also gave three conjectures on the skew-growth function for an Artin monoid of finite type.

with the classical Rodrigues's formula in the theory of orthogonal polynomials [Sz], we call the formula *Rodrigues type formula*. As a consequence of the formulae, the polynomials are expressed by a use of orthogonal polynomials. In §3, we will show that the series of the f -polynomials of type A_l and B_l satisfies 3-term recurrence relation, respectively, and the series of the f -polynomials of type D_l satisfies 4-term recurrence relation. In §4, we will prove Conjecture 2 except for types D_l affirmatively. For type A_l (resp. B_l), the f -polynomial, up to a constant factor, coincides with a certain orthogonal polynomial. Therefore, Conjecture 2 for types A_l and B_l is true ([Sz]). For exceptional types E_6 , E_7 , E_8 , F_4 , G_2 and non-crystallographic types H_3 , H_4 and $I_2(p)$, we will construct Sturm's sequence on the interval $[0, 1]$. Due to the Sturm Theorem (see for instance [T] Theorem 3.3), Conjecture 2 for them is true. In §5, we will prove Conjecture 2 for types D_l affirmatively. By applying the intermediate value theorem to the equation (2.3), we show that Conjecture 2 for types D_l is true. In §6, we will prove Conjecture 3 affirmatively. For series A_l , due to the equation (2.1), the smallest zero locus of $f_{A_l}(t)$ coincides with the smallest zero locus of $f_{A_{l+1}}^+(t)$. From Theorem 6.1 in [I-S], we show that Conjecture 3 for series A_l is true. For series B_l , from the equation (2.9), we show that the smallest zero locus of $f_{B_l}(t)$ coincides with the smallest zero locus of the shifted Legendre polynomial $\tilde{P}_l^{(0,0)}(t) := P_l^{(0,0)}(2t-1)$ (the shifting of Legendre polynomial $P_l^{(0,0)}(t)$) of degree l . From [Sz] Theorem 6.21.3, we have that the smallest zero locus of Legendre polynomial $P_l^{(0,0)}(t)$ monotonously decreasingly converges to zero as the rank l tends to infinity. Therefore, we show that Conjecture 3 for series B_l is true. The proof for the series D_l uses again the equation (2.3), where the polynomials of type D_l are expressed by those of type B_l so that the roots of type D_l are sandwiched by the roots of type B_l . Hence, we show that Conjecture 3 for types D_l is true. In §7, we will prove Conjecture 4 except for types D_l affirmatively. For series A_l , due to the equation (2.1), we have that Conjecture 4 for series A_l is true. Since the f -polynomial of type B_l , up to a constant factor, coincides with the shifted Legendre polynomial $\tilde{P}_l^{(0,0)}(t)$, due to Proposition 6.6 in [I-S], we have that Conjecture 4 for series B_l is true. In §8, we will prove Conjecture 4 for types D_l affirmatively. The proof is divided into two parts and is more complicated. In Part I, we discuss location of the following roots $t_{D_l, l+1-\nu}$ and $t_{D_l, l+1-\nu}^+$ ($\nu = 1, \dots, \lfloor l/2 \rfloor$). In Part II, we discuss location of the following roots $t_{D_l, \lfloor l/2 \rfloor + 1 - \nu}$ and $t_{D_l, \lfloor l/2 \rfloor + 1 - \nu}^+$ ($\nu = 1, \dots, \lceil l/2 \rceil$).

Remark 1.2. In [Br], the author showed that the h -polynomial of type P has l simple real roots on the interval $(-\infty, 0)$. The h -polynomial of type P in the formal variable t is defined as

$$h_P(t) := \sum_{i=0}^l h_i(P) t^i,$$

where the coefficients $h_i(P)$ are defined by the equation

$$\sum_{i=0}^l f_{i-1}(P)(x-1)^{l-i} = \sum_{i=0}^l h_i(P)x^{l-i}.$$

Hence, we have

$$(1.1) \quad f_P(t) = (1-t)^l h_P\left(\frac{t}{t-1}\right).$$

From (1.1), this implies that Conjecture 2. is approved.

2. RODRIGUES TYPE FORMULAE AND ORTHOGONAL POLYNOMIALS

In this section, we first prepare some useful propositions for later use. Next, for the three infinite series A_l , B_l and D_l of f -polynomials, we show that they are expressed in higher (logarithmic) derivatives of the polynomials of the form $t^p(1-t)^q$. In analogy with the classical Rodrigues's formula [Sz], we call the formula *Rodrigues type formula*. As a consequence of the formulae, the polynomials are expressed by a use of orthogonal polynomials.

Proposition 2.1. 1. For type A_l , the following identity holds for $l = 1, 2, \dots$:

$$(2.1) \quad (1-t)f_{A_l}(t) = f_{A_{l+1}}^+(t).$$

2. For type B_l , the following identity holds for $l = 2, 3, \dots$:

$$(2.2) \quad f_{B_l}(t) + f_{B_{l-1}}(t) = 2f_{B_l}^+(t).$$

3. The following identity holds for $l = 4, 5, \dots$:

$$(2.3) \quad f_{D_l}(t) = \frac{l-2}{2(l-1)}f_{B_l}(t) + \frac{l}{2(l-1)}(1-2t)f_{B_{l-1}}(t).$$

Proof. 1: From Table A, we compute the coefficient of $(-t)^k$ of $(1-t)f_{A_l}(t)$.

$$\frac{1}{l+2} \binom{l}{k} \binom{l+k+2}{k+1} + \frac{1}{l+2} \binom{l}{k-1} \binom{l+k+1}{k} = \frac{1}{l+1} \binom{l+1}{k} \binom{l+k+1}{k+1}.$$

This coincides with the coefficient of $(-t)^k$ of $N_{G_{A_{l+1}}^{dual}}(t)$ in Table A in [I-S].

2: From Table A, we compute the coefficient of $(-t)^k$ on the LHS of (2.2).

$$\binom{l}{k} \binom{l+k}{k} + \binom{l-1}{k} \binom{l+k-1}{k} = 2 \binom{l}{k} \binom{l+k-1}{k}.$$

This coincides with the coefficient of $(-t)^k$ of $2f_{B_l}^+(t) (= 2N_{G_{B_l}^{dual}}(t))$ in Table A in [I-S].

3: From Table A, we compute the coefficient of $(-t)^k$ on the RHS of (2.3).

$$\begin{aligned} & \frac{l-2}{2(l-1)} \cdot \frac{(l+k)!}{(l-k)!k!k!} + \frac{l}{2(l-1)} \cdot \frac{(l+k-1)!}{(l-1-k)!k!k!} + \frac{l(l+k-2)!}{(l-1)(l-k)!(k-1)!(k-1)!} \\ &= \frac{(l+k-2)!}{(l-k)!k!k!} \left\{ l(l+k-1) + k(k-1) \right\}. \end{aligned}$$

This coincides with the coefficient of $(-t)^k$ on the left hand side in Table A. \square

Theorem 2.2. (Rodrigues type formula) For types A_l ($l \geq 1$), B_l ($l \geq 2$) and D_l ($l \geq 4$), we have the formulae:

$$(2.4) \quad t(1-t)f_{A_l}(t) = \frac{1}{(l+1)!} \frac{d^l}{dt^l} \left[t^{l+1}(1-t)^{l+1} \right],$$

$$(2.5) \quad f_{B_l}(t) = \frac{1}{l!} \frac{d^l}{dt^l} \left[t^l(1-t)^l \right],$$

$$(2.6) \quad f_{D_l}(t) = \frac{1}{(l-1)!} \frac{d^{l-1}}{dt^{l-1}} \left[t^{l-1}(1-t)^l \right] + \frac{1}{(l-2)!} \frac{d^{l-2}}{dt^{l-2}} \left[t^l(1-t)^{l-2} \right]$$

$$(2.7) \quad = \frac{1}{(l-1)!} \frac{d^{l-2}}{dt^{l-2}} \left[t^{l-2}(1-t)^{l-2} \left\{ (l-1) - (3l-2)t + (3l-2)t^2 \right\} \right].$$

Proof. Type A_l : Due to Theorem 2.1 in [I-S], the right hand side of (2.4) coincides with $tf_{A_{l+1}}^+(t)$. Thanks to (2.1), this coincides with the left hand side of (2.4).

Type B_l : The right hand side of (2.5) is calculated as

$$\frac{1}{l!} \frac{d^l}{dt^l} \left[t^l(1-t)^l \right] = \frac{1}{l!} \frac{d^l}{dt^l} \left[\sum_{k=0}^l (-1)^k \binom{l}{k} t^{l+k} \right] = \sum_{k=0}^l (-1)^k \frac{(l+k)!}{(l-k)!k!k!} t^k.$$

This gives RHS of the expression of $f_{B_l}(t)$ in Table A.

Type D_l : We compute the right hand side of (2.6).

$$\begin{aligned} & \frac{1}{(l-1)!} \frac{d^{l-1}}{dt^{l-1}} \left[\sum_{k=0}^l (-1)^k \binom{l}{k} t^{l+k-1} \right] + \frac{1}{(l-2)!} \frac{d^{l-2}}{dt^{l-2}} \left[\sum_{k=0}^{l-2} (-1)^k \binom{l-2}{k} t^{l+k} \right] \\ &= \sum_{k=0}^l (-1)^k \frac{l(l+k-1)!}{(l-k)!k!k!} t^k + \sum_{k=0}^{l-2} (-1)^k \frac{(l+k)!}{(l-2-k)!k!(k+2)!} t^{k+2} \\ &= \sum_{k=0}^l (-1)^k \frac{l(l+k-1)!}{(l-k)!k!k!} t^k + \sum_{k=2}^l (-1)^k \frac{(l+k-2)!}{(l-k)!k!(k-2)!} t^k \\ &= \sum_{k=0}^l (-1)^k \left(\binom{l}{k} \binom{l+k-1}{k} + \binom{l-2}{k-2} \binom{l+k-2}{k} \right) t^k. \end{aligned}$$

This gives RHS of the expression of $f_{D_l}(t)$ in Table A. \square

For $l \in \mathbb{Z}_{\geq 0}$ and $\alpha, \beta \in \mathbb{R}_{>-1}$, let $P_l^{(\alpha, \beta)}(x)$ be the Jacobi polynomial (c.f. [Sz] 2.4). Let us introduce the *shifted Jacobi polynomial* of degree l by setting

$$\tilde{P}_l^{(\alpha, \beta)}(t) := P_l^{(\alpha, \beta)}(2t - 1).$$

Fact 2.3. [Sz](4.3.1) *The shifted Jacobi polynomial satisfies the following equality*

$$(t - 1)^\alpha t^\beta \tilde{P}_l^{(\alpha, \beta)}(t) = \frac{1}{l!} \frac{d^l}{dt^l} [(t - 1)^{l+\alpha} t^{l+\beta}].$$

Comparing two formulae in Theorem 2.2 and Fact 2.3, we obtain expression of the f -polynomials for types A_l , B_l and D_l by shifted Jacobi polynomials.

$$(2.8) \quad f_{A_l}(t) = \frac{(-1)^l}{l+1} \tilde{P}_l^{(1,1)}(t),$$

$$(2.9) \quad f_{B_l}(t) = (-1)^l \tilde{P}_l^{(0,0)}(t),$$

$$(2.10) \quad f_{D_l}(t) = (-1)^{l-1} (1-t) \tilde{P}_{l-1}^{(1,0)}(t) + (-1)^l t^2 \tilde{P}_{l-2}^{(0,2)}(t).$$

Remark 2.4. *From (2.3), we obtain the following expression of the f -polynomial for type D_l by shifted Legendre polynomials.*

$$(2.11) \quad (-1)^l f_{D_l}(t) = \frac{l-2}{2(l-1)} \tilde{P}_l^{(0,0)}(t) - \frac{l}{2(l-1)} (1-2t) \tilde{P}_{l-1}^{(0,0)}(t).$$

3. RECURRENCE RELATIONS FOR TYPES A_l ($l \geq 1$), B_l ($l \geq 2$) AND D_l ($l \geq 4$)

As an application of the Rodrigues type formulae, we show that the series of f -polynomials for types A_l ($l \geq 1$), B_l ($l \geq 2$) and D_l satisfy either 3-term or 4-term recurrence relations (Theorem 3.1).

Theorem 3.1. *For type A_l and B_l , the following 3-term recurrence relation holds.*

$$(3.1) \quad \begin{aligned} (l+4)f_{A_{l+2}}(t) &= (2l+5)(1-2t)f_{A_{l+1}}(t) - (l+1)f_{A_l}(t), \\ (l+2)f_{B_{l+2}}(t) &= (2l+3)(1-2t)f_{B_{l+1}}(t) - (l+1)f_{B_l}(t). \end{aligned}$$

For type D_l , the following 4-term recurrence relation holds.

$$(3.2) \quad f_{D_{l+3}}(t) = (a_l + b_l t) f_{D_{l+2}}(t) + (c_l + d_l t + e_l t^2) f_{D_{l+1}}(t) + (f_l + g_l t) f_{D_l}(t).$$

Here, a_l , b_l , c_l , d_l , e_l , f_l and g_l are the following rational functions:

$$a_l = \frac{(l+1)(5l^2 + 4l - 21)}{(l-1)(l+3)(5l+4)},$$

$$b_l = -\frac{2(l+1)(5l^2 + 4l - 21)}{(l-1)(l+3)(5l+4)},$$

$$c_l = \frac{l(5l^2 + 14l + 5)}{(l-1)(l+3)(5l+4)},$$

$$d_l = -\frac{4l(2l+1)(5l+9)}{(l-1)(l+3)(5l+4)},$$

$$e_l = \frac{4l(2l+1)(5l+9)}{(l-1)(l+3)(5l+4)},$$

$$f_l = -\frac{(l+1)(5l+9)}{(l+3)(5l+4)},$$

$$g_l = \frac{2(l+1)(5l+9)}{(l+3)(5l+4)}.$$

Proof. Let us consider the k th coefficient of $f_{D_l}(t)$ up to the sign $(-1)^k$:

$$\begin{aligned} \mathcal{C}(l, k) &:= \binom{l}{k} \binom{l+k-1}{k} + \binom{l-2}{k-2} \binom{l+k-2}{k} \\ &= \frac{(l+k-2)!}{(l-k)!(k!)^2} \{l(l+k-1) + k(k-1)\}. \end{aligned}$$

We compute the coefficient of the term $(-t)^k$ on the right hand side of (3.2).

$$\begin{aligned} &a_l \cdot \mathcal{C}(l+2, k) - b_l \cdot \mathcal{C}(l+2, k-1) + c_l \cdot \mathcal{C}(l+1, k) - d_l \cdot \mathcal{C}(l+1, k-1) \\ &+ e_l \cdot \mathcal{C}(l+1, k-2) + f_l \cdot \mathcal{C}(l, k) - g_l \cdot \mathcal{C}(l, k-1) \\ &= \frac{(l+k-3)!}{(l+3-k)!(k!)^2(l-1)(l+3)(5l+4)} \left[(l+1)(5l^2+4l-21)(l+k)(l+k-1)(l+k-2) \right. \\ &\quad \times (l-k+3)\{(l+2)(l+k+1) + k(k-1)\} \\ &\quad + 2(l+1)(5l^2+4l-21)k^2(l+k-1)(l+k-2) \\ &\quad \times \{(l+2)(l+k) + (k-1)(k-2)\} \\ &\quad + l(5l^2+4l+5)(l-k+3)(l-k+2)(l+k-1)(l+k-2) \\ &\quad \times \{(l+1)(l+k) + k(k-1)\} \\ &\quad + 4l(2l+1)(5l+9)k^2(l+k-2)(l-k+3) \\ &\quad \times \{(l+1)(l+k-1) + (k-1)(k-2)\} \\ &\quad + 4l(2l+1)(5l+9)k^2(k-1)^2 \\ &\quad \times \{(l+1)(l+k-2) + (k-2)(k-3)\} \\ &\quad - (l-1)(l+1)(5l+9)(l+k-2)(l-k+3)(l-k+2)(l-k+1) \\ &\quad \times \{l(l+k-1) + k(k-1)\} \\ &\quad \left. - 2(l-1)(l+1)(5l+9)(l-k+3)(l-k+2)k^2 \right. \\ &\quad \left. \times \{l(l+k-2) + (k-1)(k-2)\} \right] \\ &= \frac{(l+k+1)!}{(l+3-k)!(k!)^2} (6+2k+k^2+5l+kl+l^2) \\ &= \mathcal{C}(l+3, k). \end{aligned}$$

□

As an application of the recurrence relation, we observe the following.

Corollary 3.2. *For each types $P_l = A_l (l \geq 1), B_l (l \geq 2)$, the f -polynomial $f_{P_l}(t)$ is divisible by $2t-1$ if and only if l is odd.*

Therefore, due to (2.3), we obtain the following.

Remark 3.3. *The f -polynomial $f_{D_l}(t)$ ($l \geq 4$) is divisible by $2t-1$ if and only if l is odd.*

Although for each types $P_l = A_l (l \geq 1), B_l (l \geq 2)$ the f -polynomial $f_{P_l}(t)$ is a solution of the Gauss hypergeometric differential equation, the f -polynomial $f_{D_l}(t)$ is a solution of the following Fuchsian equation of third-order.

Remark 3.4. *The f -polynomial $f_{D_l}(t)$ satisfies the following Fuchsian ordinary differential equation of third-order. The proof is left to the reader.*

$$t(t-1)(2t-1) \frac{d^3 y}{dt^3} + \{(l+6)(t^2-t) + 2\} \frac{d^2 y}{dt^2} - l(l-1)(2t-1) \frac{dy}{dt} - l(l-1)(l+2)y = 0.$$

4. PROOF OF CONJECTURE 2 EXCEPT FOR TYPES D_l

In the present section, we prove, except for types D_l , the following theorem, which approves Conjecture 2. The proof for types D_l will be given in the next section 5.

Theorem 4.1. *The f -polynomial $f_P(t)$ for any finite type P has $\text{rank}(P)$ simple roots on the interval $(0, 1)$.*

Proof. **Case I: type A_l ($l \in \mathbb{Z}_{\geq 1}$) and B_l ($l \in \mathbb{Z}_{\geq 1}$).**

This is an immediate consequence of the formulae (2.8) and (2.9), since the Jacobi polynomials $\tilde{P}_l^{(1,1)}$ and $\tilde{P}_l^{(0,0)}$ are well known to have l simple roots on the interval $(0, 1)$ (see [Sz] Theorem 3.3.1).

Case II: Exceptional types and non-crystallographic types

We apply the Euclid division algorithm for the pair of polynomials $f_0 := f_P$ and $f_1 := f'_P$. So, we obtain, a sequence f_0, f_1, f_2, \dots of polynomials in t such that $f_{k-1} = f_k \cdot q_{k-1} + f_{k+1}$ for $k = 1, 2, \dots$ (where q_{k-1} is the quotient and f_{k+1} is the remainder).

Then, we prove the following fact by direct calculations case by case.

Fact 4.2. *i) The degrees of the sequence f_0, f_1, f_2, \dots of polynomials descend one by one, and f_l is a non-zero constant.*

ii) The sequence $f_0(0), f_1(0), -f_2(0), \dots, (-1)^{l-1}f_l(0)$ has constant sign and the sequence $f_0(1), f_1(1), -f_2(1), \dots, (-1)^{l-1}f_l(1)$ has alternating sign.

Applying the Sturm theorem (see for instance [T] Theorem 3.1), we observe that f_0 has l distinct roots on the interval $(0, 1)$.

This completes a proof of Theorem 4.1. \square

5. PROOF OF CONJECTURE 2 FOR TYPES D_l ($l \geq 4$)

In this section, we prove the following theorem, which answers to Conjecture 2 for the types D_l ($l \geq 4$) affirmatively.

Theorem 5.1. *The polynomial $f_{D_l}(t)$ has l simple roots on the interval $(0, 1)$.*

Proof. First, we recall that the shifted Legendre polynomial $\tilde{P}_l^{(0,0)}(t) = (-1)^l f_{B_l}(t)$ satisfies the functional equation $\tilde{P}_l^{(0,0)}(t) = (-1)^l \tilde{P}_l^{(0,0)}(1-t)$. Hence, thanks to (2.3),

Lemma 5.2. $f_{D_l}(t) = (-1)^l f_{D_l}(1-t)$.

Secondly, let $t_{B_l, \nu}, \nu = 1, 2, \dots, l$, be the zeros of $f_{B_l}(t)$ in decreasing order (i.e. $1 > t_{B_l, 1} > t_{B_l, 2} > \dots > t_{B_l, l} > 0$). Then, the following fact is known (see [Sz] Theorem 3.3.2.).

Fact 5.3. *The system $\{t_{B_l, \nu}\}_{\nu=1}^l$ alternates with the system $\{t_{B_{l+1}, \nu}\}_{\nu=1}^{l+1}$, that is,*

$$t_{B_{l+1}, \nu} > t_{B_l, \nu} > t_{B_{l+1}, \nu+1}, \quad (\nu = 1, \dots, l).$$

Case I: $l = 2k$

We consider $2k$ open intervals $(t_{B_{2k}, 2k+1-\nu}, t_{B_{2k-1}, 2k-\nu})(\nu = 1, 2, \dots, k)$ and $(t_{B_{2k-1}, k+1-\nu}, t_{B_{2k}, k+1-\nu})(\nu = 1, 2, \dots, k)$. We note that $t_{B_{2k-1}, k} = 1/2$. On the intervals $(t_{B_{2k}, 2k+1-\nu}, t_{B_{2k-1}, 2k-\nu})(\nu = 1, 2, \dots, k)$, the polynomials $f_{B_{2k}}(t)$ and $f_{B_{2k-1}}(t)$ have the opposite sign. Moreover, due to the identity (2.3), we can show

$f_{D_{2k}}(t_{B_{2k}, 2k+1-\nu}) f_{D_{2k}}(t_{B_{2k-1}, 2k-\nu}) < 0, \nu = 1, \dots, k$. Thanks to intermediate value theorem, for each interval $(t_{B_{2k}, 2k+1-\nu}, t_{B_{2k-1}, 2k-\nu})(\nu = 1, 2, \dots, k)$, there exists at least one root of $f_{D_{2k}}(t)$. Due to Lemma 5.2, we have that for each interval $(t_{B_{2k-1}, k+1-\nu}, t_{B_{2k}, k+1-\nu})(\nu = 1, 2, \dots, k)$, there exists at least one root of $f_{D_{2k}}(t)$. Since the polynomial $f_{D_{2k}}(t)$ is of precise degree $2k$, we conclude that, in each intervals $(t_{B_{2k}, 2k+1-\nu}, t_{B_{2k-1}, 2k-\nu})(\nu = 1, 2, \dots, k)$ and $(t_{B_{2k-1}, k+1-\nu}, t_{B_{2k}, k+1-\nu})(\nu = 1, 2, \dots, k)$, there is one and only one root of the polynomial $f_{D_{2k}}(t)$.

Case II: $l = 2k + 1$

In a similar manner to Case I, we conclude that the polynomial $f_{D_l}(t)$ has l simple roots on the interval $(0, 1)$. \square

From the discussion in the proof of Theorem 5.1, we have shown the following.

Corollary 5.4. *1. For the case where $l = 2k$, we obtain the following properties for $\nu = 1, 2, \dots, k$:*

$$\begin{aligned} t_{D_{2k}, 2k+1-\nu} &\in (t_{B_{2k}, 2k+1-\nu}, t_{B_{2k-1}, 2k-\nu}), \\ t_{D_{2k}, k+1-\nu} &\in (t_{B_{2k-1}, k+1-\nu}, t_{B_{2k}, k+1-\nu}). \end{aligned}$$

2. For the case where $l = 2k + 1$, we obtain the following properties for $\nu = 1, 2, \dots, k$:

$$\begin{aligned} t_{D_{2k+1}, 2k+2-\nu} &\in (t_{B_{2k+1}, 2k+2-\nu}, t_{B_{2k}, 2k+1-\nu}), \quad t_{D_{2k+1}, k+1} = 1/2, \\ t_{D_{2k+1}, k+1-\nu} &\in (t_{B_{2k}, k+1-\nu}, t_{B_{2k+1}, k+1-\nu}). \end{aligned}$$

Remark 5.5. For an irreducible (possibly noncrystallographic) root system of rank l of type P , F. Chapoton defined Chapoton's F -triangle⁵ ([C]) of type P as

$$(5.1) \quad F_P(x, y) := \sum_{k=0}^l \sum_{m=0}^l f_{k,m}(-x)^k (-y)^m$$

, where $f_{k,m}$ is the number of faces of the cluster complex $\Delta(P)$ consisting of k positive roots and m negative simple roots. Clearly, $f_{k,m} = 0$ unless $k + m \leq l$. We note that $f_P^+(x) = F_P(x, 0)$ and $f_P(x) = F_P(x, x)$. In [Sa4], the author studied the zero locus of Chapoton's F -triangle. Based on some numerical experiments, the author gave some conjectures on the zeros of F -triangle.

6. PROOF OF CONJECTURE 3

In this section, we prove the following theorem, which approves Conjecture 3. Let us fix notation: for $P \in \{A_l \ (l \geq 1), B_l \ (l \geq 1), D_l \ (l \geq 4), E_l \ (l = 6, 7, 8), F_4, G_2, H_3, H_4, I_2(p) \ (p \geq 3)\}$, let $t_{P,\nu}, \nu = 1, 2, \dots, l = \text{rank}(P)$, be the zeros of $f_P(t)$ in decreasing order (i.e. $1 > t_{P,1} > t_{P,2} > \dots > t_{P,l} > 0$).

Theorem 6.1. For each series of types $P_l = A_l, B_l, D_l$, the smallest zero locus of $f_{P_l}(t)$ monotonously decreasingly converges to zero as the rank l tends to infinity.

Proof. Type A_l : Due to Theorem 6.1 in [I-S], the smallest zero locus of $f_{A_{l+1}}^+(t)$ monotonously decreasingly converges to zero as the rank l tends to infinity. Hence, from (2.1), we show that the smallest zero locus of $f_{A_l}(t)$ monotonously decreasingly converges to zero. Type B_l : Recall a fact on the distribution of the zeros of $\tilde{P}_l^{(0,0)}(t) = (-1)^l f_{B_l}(t)$ ([Sz] Theorem 6.21.3).

Fact 6.2. Let $\tilde{x}_\nu = \tilde{x}_{l,\nu}, \nu = 1, 2, \dots, l$, be the zeros of $\tilde{P}_l^{(0,0)}(t)$ in decreasing order. Let $\theta_l = \theta_{l,\nu} \in (0, \pi), \nu = 1, 2, \dots, l$, be the real number defined by

$$\cos \theta_\nu = 2\tilde{x}_\nu - 1.$$

Then, the inequalities hold as follows: $\frac{\nu - \frac{1}{2}}{l + \frac{1}{2}}\pi < \theta_\nu < \frac{\nu}{l + \frac{1}{2}}\pi \quad (\nu = 1, 2, \dots, l)$.

Therefore, we have that the smallest zero locus of $f_{B_l}(t)$ monotonously decreasingly converges to zero.

Type D_l : Due to Corollary 5.4, we show that the smallest zero locus $t_{D_l,l}$ of $f_{D_l}(t)$ is an element of the interval $(t_{B_l,l}, t_{B_{l-1},l-1})$. Therefore, we conclude that the smallest zero locus $t_{D_l,l}$ monotonously decreasingly converges to zero. \square

7. PROOF OF CONJECTURE 4 EXCEPT FOR TYPES D_l

In this section, we prove, except for types D_l , the following theorem, which approves Conjecture 4. The proof for types D_l will be given in the next section 8. Let us fix notation: for $P \in \{A_l \ (l \geq 1), B_l \ (l \geq 1), D_l \ (l \geq 4), E_l \ (l = 6, 7, 8), F_4, G_2, H_3, H_4, I_2(p) \ (p \geq 3)\}$, let $t_{P,\nu}^+, \nu = 1, 2, \dots, l = \text{rank}(P)$, be the zeros of $f_P^+(t)$ in decreasing order (i.e. $1 = t_{P,1}^+ > t_{P,2}^+ > \dots > t_{P,l}^+ > 0$).

Theorem 7.1. The following inequalities hold for any type P

$$t_{P,\nu} > t_{P,\nu+1}^+ > t_{P,\nu+1}, \quad (\nu = 1, \dots, l-1).$$

Proof. I. Case for types A_l and B_l .

First, for type A_l , we recall the following fact (see [Sz] Theorem 3.3.2.).

⁵In the definition, the sign of variables changes from the original definition by F. Chapoton.

Fact 7.2. *The system $\{t_{A_l, \nu}^+\}_{\nu=2}^l$ alternates with the system $\{t_{A_{l+1}, \nu}^+\}_{\nu=2}^{l+1}$, that is,*

$$t_{A_{l+1}, \nu}^+ > t_{A_l, \nu}^+ > t_{A_{l+1}, \nu+1}^+, \quad (\nu = 2, \dots, l).$$

Hence, from (2.1), we have the following inequalities

$$t_{A_l, \nu}^+ > t_{A_l, \nu+1}^+ > t_{A_l, \nu+1}, \quad (\nu = 1, \dots, l-1).$$

Next, we recall $f_{B_l}(t) = (-1)^l \tilde{P}_l^{(0,0)}(t)$. Due to Proposition 6.6 in [I-S], we have the following inequalities

$$t_{B_l, \nu}^+ > t_{B_l, \nu+1}^+ > t_{B_l, \nu+1}, \quad (\nu = 1, \dots, l-1).$$

II. Exceptional types and non-crystallographic types.

In section 4, for $P \in \{E_l \ (l = 6, 7, 8), F_4, G_2, H_3, H_4, I_2(p) \ (p \geq 3)\}$, we constructed a Sturm sequence $f_0(t), f_1(t), -f_2(t), \dots, (-1)^{l-1} f_l(t)$ on $[0, 1]$. In [I-S]§4, for the pair of polynomials $f_0^+ := f_P^+(t)/(1-t)$ and $f_1^+ := (f_0)^+$, we constructed a Sturm sequence $f_0^+(t), f_1^+(t), -f_2^+(t), \dots, (-1)^{l-2} f_{l-1}^+(t)$ on $[0, 1]$. For $t_0 \in [0, 1]$, let $V(f_P, t_0)$ (resp. $V(f_P^+, t_0)$) be the number of sign changes in the sequence $f_0(t), f_1(t), -f_2(t), \dots, (-1)^{l-1} f_l(t)$ (resp. $f_0^+(t), f_1^+(t), -f_2^+(t), \dots, (-1)^{l-2} f_{l-1}^+(t)$). For $t_0 \in [0, 1]$, we put

$$\overline{V}(f_P, t_0) := V(f_P, 0) - V(f_P, t_0), \quad \overline{V}(f_P^+, t_0) := V(f_P^+, 0) - V(f_P^+, t_0).$$

Then, we prove the following fact case by case.

Fact 7.3. *There exist sequences $\{\alpha_{P, \nu}\}_{\nu=1}^{l-1}$ and $\{\alpha_{P, \nu}^+\}_{\nu=1}^{l-1}$ of real numbers which satisfy inequalities $0 < \alpha_{P, l-1} < \alpha_{P, l-1}^+ < \dots < \alpha_{P, 1} < \alpha_{P, 1}^+ < 1$ such that*

$$\overline{V}(f_P, \alpha_{P, i}) = l - i, \quad \overline{V}(f_P, \alpha_{P, i}^+) = l - i,$$

$$\overline{V}(f_P^+, \alpha_{P, i}) = l - i - 1, \quad \overline{V}(f_P^+, \alpha_{P, i}^+) = l - i, \quad (i = 1, \dots, l-1).$$

Therefore, due to the Sturm theorem (see for instance [T] Theorem 3.1), we have that $t_{P, \nu} \in (\alpha_{P, \nu}^+, \alpha_{P, \nu-1})$ and $t_{P, \nu}^+ \in (\alpha_{P, \nu-1}, \alpha_{P, \nu-1}^+)$ ($\nu = 2, \dots, l$), where we put $\alpha_{P, l}^+ := 0$. Hence, we have the following inequalities

$$t_{P, \nu} > t_{P, \nu+1}^+ > t_{P, \nu+1}, \quad (\nu = 1, \dots, l-1).$$

□

Example For each $P \in \{E_l \ (l = 6, 7, 8), F_4, G_2, H_3, H_4, I_2(p) \ (p \geq 3)\}$, we give an example of two kinds of sequences $\{\alpha_{P, \nu}\}_{\nu=1}^{l-1}$ and $\{\alpha_{P, \nu}^+\}_{\nu=1}^{l-1}$ that satisfy the inequalities in Fact 7.3.

E_6 :

$$\alpha_{E_6, 5} = 7/200, \alpha_{E_6, 5}^+ = 1/10, \alpha_{E_6, 4} = 21/100, \alpha_{E_6, 4}^+ = 1/4, \alpha_{E_6, 3} = 2/5,$$

$$\alpha_{E_6, 3}^+ = 3/5, \alpha_{E_6, 2} = 13/20, \alpha_{E_6, 2}^+ = 7/10, \alpha_{E_6, 1} = 17/20, \alpha_{E_6, 1}^+ = 9/10.$$

E_7 :

$$\alpha_{E_7, 6} = 1/50, \alpha_{E_7, 6}^+ = 1/10, \alpha_{E_7, 5} = 3/20, \alpha_{E_7, 5}^+ = 1/5, \alpha_{E_7, 4} = 8/25,$$

$$\alpha_{E_7, 4}^+ = 2/5, \alpha_{E_7, 3} = 13/25, \alpha_{E_7, 3}^+ = 3/5, \alpha_{E_7, 2} = 7/10, \alpha_{E_7, 2}^+ = 4/5,$$

$$\alpha_{E_7, 1}^+ = 87/100, \alpha_{E_7, 1} = 19/20.$$

E_8 :

$$\alpha_{E_8, 7} = 19/2000, \alpha_{E_8, 7}^+ = 1/100, \alpha_{E_8, 6} = 11/100, \alpha_{E_8, 6}^+ = 1/5, \alpha_{E_8, 5} = 1/4,$$

$$\alpha_{E_8, 5}^+ = 3/10, \alpha_{E_8, 4} = 21/50, \alpha_{E_8, 4}^+ = 49/100, \alpha_{E_8, 3} = 3/5, \alpha_{E_8, 3}^+ = 7/10,$$

$$\alpha_{E_8, 2} = 77/100, \alpha_{E_8, 2}^+ = 4/5, \alpha_{E_8, 1} = 9/10, \alpha_{E_8, 1}^+ = 19/20.$$

F_4 :

$$\alpha_{F_4, 3} = 1/20, \alpha_{F_4, 3}^+ = 1/10, \alpha_{F_4, 2} = 7/20, \alpha_{F_4, 2}^+ = 2/5, \alpha_{F_4, 1} = 7/10, \alpha_{F_4, 1}^+ = 4/5.$$

$$G_2 : \alpha_{G_2, 1} = 1/6, \alpha_{G_2, 1}^+ = 1/2.$$

$$H_3 : \alpha_{H_3, 2} = 7/100, \alpha_{H_3, 2}^+ = 1/10, \alpha_{H_3, 1} = 11/20, \alpha_{H_3, 1}^+ = 3/5.$$

$$H_4 : \alpha_{H_4, 3} = 9/500, \alpha_{H_4, 3}^+ = 1/5, \alpha_{H_4, 2} = 31/100, \alpha_{H_4, 2}^+ = 2/5, \alpha_{H_4, 1} = 7/10,$$

$$\alpha_{H_4, 1}^+ = 4/5.$$

$$I_2(p \geq 3): \alpha_{I_2(p),1} = 1/p, \alpha_{I_2(p),1}^+ = 1/2.$$

8. PROOF OF CONJECTURE 4 FOR TYPES D_l ($l \geq 4$)

In the case where $l = 4$, we approve Conjecture 4 by hand calculation. Throughout this section, we assume $l \geq 5$.

Theorem 8.1. *The following inequalities hold for $\nu = 1, \dots, l-1$*

$$t_{D_l, \nu} > t_{D_l, \nu+1}^+ > t_{D_l, \nu+1}.$$

Proof. First, we recall an equation from [I-S]§5

$$f_{D_l}^+(t) = \frac{(-1)^{l-3}}{(l-2)!} ((1-t)H_l^{(l-3)}(t) - (l-3)H_l^{(l-4)}(t)),$$

where $H_l(t) := (t^2 - t)^{l-3} \{(l-2) - (3l-4)t + (3l-4)t^2\}$ and its higher order derivatives $H_l^{(i)}(t) := \frac{d^i}{dt^i} H_l(t)$ for $0 \leq i \leq l-3$. From Lemma 5.2 in [I-S], let $1 = u_1 > u_2 > \dots > u_{l-1} > u_l = 0$ be all roots of the polynomial $H_l^{(l-4)}(t) = 0$. Let $1 > v_1 > v_2 > \dots > v_{l-1} > 0$ be the $l-1$ roots of $H_l^{(l-3)}(t) = 0$ so that one has the inequalities:

$$u_1 > v_1 > u_2 > v_2 > \dots > u_{l-1} > v_{l-1} > u_l.$$

Moreover, in the proof of Lemma 5.2 in [I-S], we have shown the following fact.

Fact 8.2. $t_{D_l, l+1-\nu}^+ \in (u_{l+1-\nu}, v_{l-\nu})$, ($\nu = 1, \dots, l-1$) and $t_{D_l, 1}^+ = 1$.

Next, the proof for Theorem 8.1 is divided into two parts.

Part I We discuss location of the following roots

$$t_{D_l, l+1-\nu} \text{ and } t_{D_l, l+1-\nu}^+ (\nu = 1, \dots, \lfloor l/2 \rfloor).$$

First, we define two kinds of functions

$$\tilde{f}_{D_l}(t) := \frac{(-1)^{l-3}}{(l-2)!} ((1-2t)H_l^{(l-3)}(t) - 2(l-3)H_l^{(l-4)}(t)),$$

$$K_l(t) := f_{D_l}^+(t) - \tilde{f}_{D_l}(t) = \frac{(-1)^{l-3}}{(l-2)!} (tH_l^{(l-3)}(t) + (l-3)H_l^{(l-4)}(t)).$$

For the polynomial $\tilde{f}_{D_l}(t)$, we have the following proposition.

Proposition 8.3. *1. The following identity holds for $l = 5, 6, \dots$:*

$$(8.1) \quad \tilde{f}_{D_l}(t) = \frac{1}{(l-2)!} \frac{d^{l-3}}{dt^{l-3}} \left[t^{l-3} (1-t)^{l-3} (1-2t) \{(l-2) - (3l-4)t + (3l-4)t^2\} \right].$$

2. The following identity holds for $l = 5, 6, \dots$:

$$(8.2) \quad \tilde{f}_{D_l}(t) = \frac{l+2}{l} f_{D_l}(t) - \frac{2}{l} f_{B_l}(t).$$

3. The following identity holds for $l = 5, 6, \dots$:

$$(8.3) \quad \tilde{f}_{D_l}(t) = (-1)^l \tilde{f}_{D_l}(1-t).$$

Proof. 1. By definition, we have

$$2f_{D_l}^+(t) = \tilde{f}_{D_l}(t) + \frac{(-1)^{l-3}}{(l-2)!} H_l^{(l-3)}(t).$$

We recall the Rodrigues type formula for $f_{D_l}^+(t)$ from Theorem 5.1 in [I-S]

$$f_{D_l}^+(t) = \frac{1}{(l-2)!} \frac{d^{l-3}}{dt^{l-3}} \left[t^{l-3} (1-t)^{l-2} \{(l-2) - (3l-4)t + (3l-4)t^2\} \right].$$

By calculating $2f_{D_l}^+(t) - \frac{(-1)^{l-3}}{(l-2)!} H_l^{(l-3)}(t)$, we obtain the result.

2. From (2.7) and (8.1), we have

$$\tilde{f}_{D_l}(t) - f_{D_l}(t)$$

$$\begin{aligned}
&= \frac{2}{(l-1)!} \frac{d^{l-2}}{dt^{l-2}} \left[t^{l-1} (1-t)^{l-1} \right] \\
&= \frac{2}{l} f_{D_l}(t) - \frac{2}{l} f_{B_l}(t).
\end{aligned}$$

This completes the proof.

3. Since $f_{B_l}(t) = (-1)^l f_{B_l}(1-t)$ and $f_{D_l}(t) = (-1)^l f_{D_l}(1-t)$, we obtain the result. \square

Remark 8.4. As a consequence of (8.2), the polynomial $\tilde{f}_{D_l}(t)$ is divisible by $2t-1$ if and only if l is odd.

We recall a formula from [I-S]§5.

Formula B.⁶ Set $h_k^{(i)}(t) := \left(\frac{d}{dt}\right)^i (t(t-1))^k$ for $0 \leq i \leq k$. Then, we have

$$\begin{aligned}
h_k^{(2i-1)}\left(\frac{1}{2}\right) &= 0 \quad (i = 1, \dots, k), \\
h_k^{(2i)}\left(\frac{1}{2}\right) &= \left(-\frac{1}{4}\right)^{k-i} \frac{k!(2i)!}{(k-i)!i!} \quad (i = 1, \dots, k).
\end{aligned}$$

Proof of Formula B. By induction on i , we obtain the following equations

$$(8.4) \quad h_k^{(2i-1)}(t) = \sum_{j=1}^i \frac{k!(2i-1)!}{(i-j)!(2j-1)!(k-i-j+1)!} (t^2-t)^{k-i-j+1} (2t-1)^{2j-1}$$

$$(8.5) \quad h_k^{(2i)}(t) = \sum_{j=0}^i \frac{k!(2i)!}{(i-j)!(2j)!(k-i-j)!} (t^2-t)^{k-i-j} (2t-1)^{2j}.$$

These equations hold for $0 \leq i \leq k$.

From (8.4) and (8.5), we obtain the following formula.

Formula C. We have

$$h_{l-3}^{(l-3)}(0) = (-1)^{l-1} (l-3)!, \quad h_{l-2}^{(l-3)}(0) = 0.$$

Proposition 8.5. The polynomial $\tilde{f}_{D_l}(t)$ has l simple roots on the interval $(0, 1)$.

Proof. For the case where $l = 2k$, we consider $2k$ open intervals $(t_{D_{2k}, 2k+1-\nu}, t_{D_{2k}, 2k-\nu})$ ($\nu = 1, 2, \dots, k-1$), $(t_{D_{2k}, k+1}, 1/2)$ and $(1/2, t_{D_{2k}, k})$, $(t_{D_{2k}, k+1-\nu}, t_{D_{2k}, k-1-\nu})$ ($\nu = 1, 2, \dots, k-1$). For a non-zero real number θ , we put

$$\text{sgn}(\theta) := \theta/|\theta|.$$

From the discussion in the proof of Theorem 5.1, we have shown the following properties

$$\text{sgn}(f_{B_{2k}}(t_{D_{2k}, 2k+1-\nu})) = (-1)^\nu, \quad (\nu = 1, 2, \dots, k).$$

Moreover, due to **Formula B**, we have

$$f_{B_{2k}}(1/2) = \left(-\frac{1}{4}\right)^k \frac{(2k)!}{k!k!} \quad \text{and} \quad f_{D_{2k}}(1/2) = \left(-\frac{1}{4}\right)^k \frac{(2k-2)(2k-2)!}{k!(k-1)!}.$$

Hence, we have

$$\text{sgn}(\tilde{f}_{D_{2k}}(1/2)) = (-1)^k.$$

Due to the identity (8.2), we can show $\tilde{f}_{D_{2k}}(t_{D_{2k}, 2k+1-\nu}) \tilde{f}_{D_{2k}}(t_{D_{2k}, 2k-\nu}) < 0$, ($\nu = 1, \dots, k-1$) and $\tilde{f}_{D_{2k}}(t_{D_{2k}, k+1}) \tilde{f}_{D_{2k}}(1/2) < 0$. Thanks to intermediate value theorem, for each interval $(t_{D_{2k}, 2k+1-\nu}, t_{D_{2k}, 2k-\nu})$ ($\nu = 1, 2, \dots, k-1$) and $(t_{D_{2k}, 2k+1}, 1/2)$, there exists at least one root of $\tilde{f}_{D_{2k}}(t)$. From (8.3), the set of roots of $\tilde{f}_{D_{2k}}(t)$ are symmetric with respect to the reflection centered at $t = 1/2$. Therefore, we have that for each interval $(t_{D_{2k}, k+1-\nu}, t_{D_{2k}, k-\nu})$ ($\nu = 1, 2, \dots, k-1$) and $(1/2, t_{D_{2k}, k})$, there exists at least one root of $\tilde{f}_{D_{2k}}(t)$. Since the polynomial $\tilde{f}_{D_{2k}}(t)$ is of precise degree $2k$, we conclude that, in each intervals $(t_{B_{2k}, 2k+1-\nu}, t_{B_{2k}, 2k-\nu})$ ($\nu = 1, 2, \dots, k-1$), $(t_{D_{2k}, 2k+1}, 1/2)$, $(1/2, t_{D_{2k}, k})$ and $(t_{B_{2k}, k+1-\nu}, t_{B_{2k}, k+1-\nu})$ ($\nu = 1, 2, \dots, k-1$), there is one and only one root of the polynomial $\tilde{f}_{D_{2k}}(t)$.

⁶ Although, in [I-S]§5, the index i for $h_k^{(2i-1)}(t)$ (resp. $h_k^{(2i)}(t)$) ranges from 1 to $\lfloor (k+1)/2 \rfloor$ (resp. $\lfloor k/2 \rfloor$), the range of the index i can be extended to $0 \leq i \leq k$.

For the case where $l = 2k + 1$, in a similar manner to case where $l = 2k$, we conclude that the polynomial $\tilde{f}_{D_l}(t)$ has l simple roots on the interval $(0, 1)$. \square

Let $\tilde{t}_{D_l, \nu}, \nu = 1, 2, \dots, l$, be the zeros of $\tilde{f}_{D_l}(t)$ in decreasing order (i.e. $1 > \tilde{t}_{D_l, 1} > \tilde{t}_{D_l, 2} > \dots > \tilde{t}_{D_l, l} > 0$).

Proposition 8.6. 1. For the case where $l = 2k$, we obtain the following properties for $\nu = 1, 2, \dots, k - 1$:

$$\tilde{t}_{D_{2k}, 2k+1-\nu} \in (t_{D_{2k}, 2k+1-\nu}, t_{D_{2k}, 2k-\nu}), \tilde{t}_{D_{2k}, k+1} \in (t_{D_{2k}, k+1}, 1/2),$$

$$\tilde{t}_{D_{2k}, k} \in (1/2, t_{D_{2k}, k}), \tilde{t}_{D_{2k}, k-\nu} \in (t_{D_{2k}, k+1-\nu}, t_{D_{2k}, k-\nu}).$$

2. For the case where $l = 2k + 1$, we obtain the following properties for $\nu = 1, 2, \dots, k - 1$:

$$\tilde{t}_{D_{2k+1}, 2k+2-\nu} \in (t_{D_{2k+1}, 2k+2-\nu}, t_{D_{2k+1}, 2k+1-\nu}), \tilde{t}_{D_{2k+1}, k+2} \in (t_{D_{2k+1}, k+2}, 1/2),$$

$$\tilde{t}_{D_{2k+1}, k+1} = 1/2,$$

$$\tilde{t}_{D_{2k+1}, k} \in (1/2, t_{D_{2k+1}, k}), \tilde{t}_{D_{2k+1}, k-\nu} \in (t_{D_{2k+1}, k+1-\nu}, t_{D_{2k+1}, k-\nu}).$$

Proposition 8.7. 1. For the case where $l = 2k$, we obtain the following properties for $\nu = 1, 2, \dots, k$:

$$\tilde{t}_{D_{2k}, 2k+1-\nu} \in (u_{2k+1-\nu}, v_{2k-\nu}).$$

2. For the case where $l = 2k + 1$, we obtain the following properties for $\nu = 1, 2, \dots, k$:

$$\tilde{t}_{D_{2k+1}, 2k+2-\nu} \in (u_{2k+2-\nu}, v_{2k+1-\nu}), \tilde{t}_{D_{2k+1}, k+1} = 1/2.$$

Proof. By applying intermediate value theorem to the polynomial

$$\tilde{f}_{D_l}(t) = \frac{(-1)^{l-3}}{(l-2)!} ((1-2t)H_l^{(l-3)}(t) - 2(l-3)H_l^{(l-4)}(t)),$$

we obtain the results. We omit details. \square

Proposition 8.8. 1. The polynomial $K_l(t)$ has l simple roots on the interval $[0, 1)$.

2. Let $t_{K_l, \nu}, \nu = 1, 2, \dots, l$, be the zeros of $K_l(t)$ in decreasing order (i.e. $1 > t_{K_l, 1} > t_{K_l, 2} > \dots > t_{K_l, l} = 0$). For $\nu = 1, 2, \dots, l - 1$, we have

$$t_{K_l, l} = 0, t_{K_l, l-\nu} \in (v_{l-\nu}, u_{l-\nu}).$$

3. $\frac{d}{dt}K_l(t)|_{t=0} = l - 2$.

Proof. 1. - 2. By applying intermediate value theorem to the polynomial $K_l(t)$, we obtain the results. We omit details.

3. By definition, we have $K'_l(0) = \frac{(-1)^{l-3}}{(l-3)!} H_l^{(l-3)}(0)$.

Due to **Formula C**, we have

$$H_l^{(l-3)}(0) = (l-2)h_{l-3}^{(l-3)}(0) + (3l-4)h_{l-2}^{(l-3)}(0) = (-1)^{l+1}(l-2)!.$$

Hence, we obtain $\frac{d}{dt}K_l(t)|_{t=0} = l - 2$. \square

Proposition 8.9. 1. For the case where $l = 2k$, we obtain the following properties for $\nu = 1, 2, \dots, k - 1$:

$$t_{D_{2k}, 2k+1-\nu}^+ \in (\tilde{t}_{D_{2k}, 2k+1-\nu}, \tilde{t}_{D_{2k}, 2k-\nu}), t_{D_{2k}, k+1}^+ \in (\tilde{t}_{D_{2k}, k+1}, 1/2).$$

2. For the case where $l = 2k + 1$, we obtain the following properties for $\nu = 1, 2, \dots, k - 1$:

$$t_{D_{2k+1}, 2k+2-\nu}^+ \in (\tilde{t}_{D_{2k+1}, 2k+2-\nu}, \tilde{t}_{D_{2k+1}, 2k+1-\nu}), t_{D_{2k+1}, k+2}^+ \in (\tilde{t}_{D_{2k+1}, k+2}, 1/2).$$

Proof. For the case where $l = 2k$, we consider k open intervals $(\tilde{t}_{D_{2k}, 2k+1-\nu}, \tilde{t}_{D_{2k}, 2k-\nu})$ ($\nu = 1, 2, \dots, k-1$), $(\tilde{t}_{D_{2k}, k+1}, 1/2)$. From Proposition 8.8, we have shown the following properties

$$\operatorname{sgn}(K_{2k}(\tilde{t}_{D_{2k}, 2k+1-\nu})) = (-1)^{\nu+1}, (\nu = 1, \dots, k).$$

Since $f_{D_{2k}}^+(t) = \tilde{f}_{D_{2k}}(t) + K_{2k}(t)$, we have

$$\operatorname{sgn}(f_{D_{2k}}^+(\tilde{t}_{D_{2k}, 2k+1-\nu})) = (-1)^{\nu+1}, (\nu = 1, \dots, k).$$

Moreover, due to **Formula B**, we have

$$f_{D_{2k}}^+(1/2) = \left(-\frac{1}{4}\right)^k \frac{(2k-4)(2k-3)!}{k!(k-2)!}.$$

From [I-S]§5, we have

$$\operatorname{sgn}(f_{D_{2k}}^+(v_{2k-\nu})) = (-1)^\nu, (\nu = 1, 2, \dots, k).$$

From Fact 8.2, we obtain the following results for $\nu = 1, 2, \dots, k-1$:

$$t_{D_{2k}, 2k+1-\nu}^+ \in (\tilde{t}_{D_{2k}, 2k+1-\nu}, v_{2k-\nu}), t_{D_{2k}, k+1}^+ \in (\tilde{t}_{D_{2k}, k+1}, 1/2).$$

From Proposition 8.7, we also have for $\nu = 1, 2, \dots, k-1$:

$$t_{D_{2k}, 2k+1-\nu}^+ \in (\tilde{t}_{D_{2k}, 2k+1-\nu}, \tilde{t}_{D_{2k}, 2k-\nu}), t_{D_{2k}, k+1}^+ \in (\tilde{t}_{D_{2k}, k+1}, 1/2).$$

For the case where $l = 2k+1$, we consider k open intervals $(\tilde{t}_{D_{2k+1}, 2k+2-\nu}, \tilde{t}_{D_{2k+1}, 2k+1-\nu})$ ($\nu = 1, 2, \dots, k-1$), $(\tilde{t}_{D_{2k+1}, k+2}, 1/2)$. From Proposition 8.8, we have shown the following properties

$$\operatorname{sgn}(K_{2k+1}(\tilde{t}_{D_{2k+1}, 2k+2-\nu})) = (-1)^{\nu+1}, (\nu = 1, \dots, k).$$

Since $f_{D_{2k+1}}^+(t) = \tilde{f}_{D_{2k+1}}(t) + K_{2k+1}(t)$, we have

$$\operatorname{sgn}(f_{D_{2k+1}}^+(\tilde{t}_{D_{2k+1}, 2k+2-\nu})) = (-1)^{\nu+1}, (\nu = 1, \dots, k).$$

Moreover, due to **Formula B**, we have

$$f_{D_{2k+1}}^+(1/2) = \left(-\frac{1}{4}\right)^k \frac{(2k-1)!}{k!(k-1)!}.$$

From [I-S]§5, we have

$$\operatorname{sgn}(f_{D_{2k+1}}^+(v_{2k+1-\nu})) = (-1)^\nu, (\nu = 1, 2, \dots, k).$$

From Fact 8.2, we obtain the following results for $\nu = 1, 2, \dots, k-1$:

$$t_{D_{2k+1}, 2k+2-\nu}^+ \in (\tilde{t}_{D_{2k+1}, 2k+2-\nu}, v_{2k+1-\nu}), t_{D_{2k+1}, k+2}^+ \in (\tilde{t}_{D_{2k+1}, k+2}, 1/2).$$

From Proposition 8.7, we also have for $\nu = 1, 2, \dots, k-1$:

$$t_{D_{2k+1}, 2k+2-\nu}^+ \in (\tilde{t}_{D_{2k+1}, 2k+2-\nu}, \tilde{t}_{D_{2k+1}, 2k+1-\nu}), t_{D_{2k+1}, k+2}^+ \in (\tilde{t}_{D_{2k+1}, k+2}, 1/2).$$

□

Lemma 8.10.

$$(8.6) \quad f_{D_l}^+(t) = \frac{1}{2(l-1)}(lt-2)f_{B_l}(t) + \frac{1}{2(l-1)}\{2(2l-1)t^2 - 5lt + 2l\}f_{B_{l-1}}(t).$$

Proof. First, we recall an equality from (6.4) in [I-S]

$$f_{D_l}^+(t) = \frac{l-2}{2l-1}f_{B_l}^+(t) + \left(\frac{l+1}{2l-1} - t\right)f_{B_{l-1}}^+(t).$$

Second, from (1.5) we have

$$f_{D_l}^+(t) = \frac{l-2}{4l-2}(f_{B_l}(t) + f_{B_{l-1}}(t)) + \frac{1}{2}\left(\frac{l+1}{2l-1} - t\right)(f_{B_{l-1}}(t) + f_{B_{l-2}}(t)).$$

Moreover, $f_{B_l}(t) = (-1)^l \tilde{P}_l^{(0,0)}(t)$ satisfies the 3-term recurrence relation

$$lf_{B_l}(t) = (2l-1)(1-2t)f_{B_{l-1}}(t) - (l-1)f_{B_{l-2}}(t).$$

Hence, we obtain

$$\frac{l-2}{4l-2}(f_{B_l}(t) + f_{B_{l-1}}(t)) + \frac{1}{2}\left(\frac{l+1}{2l-1} - t\right)(f_{B_{l-1}}(t) + f_{B_{l-2}}(t))$$

$$= \frac{1}{2(l-1)}(lt-2)f_{B_l}(t) + \frac{1}{2(l-1)}\{2(2l-1)t^2 - 5lt + 2l\}f_{B_{l-1}}(t).$$

This coincides with the right hand side of (8.6). \square

As a corollary of Lemma 8.10, we have.

Corollary 8.11. $t_{D_l, l+1-\nu}^+ \in (t_{B_l, l+1-\nu}, t_{B_l, l-\nu})$, $(\nu = 1, \dots, l-1)$, $t_{D_l, 1}^+ = 1$.

Proof. We note that the coefficient $\frac{1}{2(l-1)}\{2(2l-1)t^2 - 5lt + 2l\}$ in (8.6) takes positive values on the interval $[0, 1]$. Due to Lemma 8.10, the sequence $f_{D_l}^+(t_{B_l, l}), f_{D_l}^+(t_{B_l, l-1}), \dots, f_{D_l}^+(t_{B_l, 1})$ has alternating sign. By applying intermediate value theorem to the polynomial $f_{D_l}^+(t)$, we obtain the results. \square

Lastly, we prove the following theorem.

Proposition 8.12. *We obtain the following inequalities for $\nu = 1, 2, \dots, \lfloor l/2 \rfloor - 1$:*

$$t_{D_l, l+1-\nu}^+ < t_{D_l, l-\nu}.$$

Proof. By combining Corollary 5.4 and Corollary 8.11, we obtain the results. \square

Remark 8.13. 1. For $l = 2k$, we have $1/2 < t_{D_{2k}, k}$.

2. For $l = 2k + 1$, we have $1/2 < t_{D_{2k+1}, k+1}^+$.

Proof. 1. This is an immediate consequence of Corollary 5.4.

2. Since $t_{B_{2k+1}, k+1} = 1/2$, from Corollary 8.11, we obtain the result. \square

Part II

We discuss location of the following roots

$$t_{D_l, \lfloor l/2 \rfloor + 1 - \nu} \text{ and } t_{D_l, \lfloor l/2 \rfloor + 1 - \nu}^+ (\nu = 1, \dots, \lfloor l/2 \rfloor).$$

We recall the assumption $l \geq 5$.

Theorem 8.14. *The following inequalities hold for $\nu = 1, \dots, \lfloor l/2 \rfloor - 1$*

$$t_{D_l, \lfloor l/2 \rfloor - \nu} > t_{D_l, \lfloor l/2 \rfloor + 1 - \nu}^+ > t_{D_l, \lfloor l/2 \rfloor + 1 - \nu}.$$

Proof. First, we prepare a proposition.

Proposition 8.15. *We have the following properties:*

$$t_{D_l, \lfloor l/2 \rfloor + 1 - \nu}^+ \in (t_{B_l, \lfloor l/2 \rfloor + 1 - \nu}, t_{B_{l-1}, \lfloor l/2 \rfloor - \nu}), (\nu = 1, \dots, \lfloor l/2 \rfloor - 1).$$

Proof. First, from corollary 8.11, we obtain

$$t_{D_l, \lfloor l/2 \rfloor + 1 - \nu}^+ \in (t_{B_l, \lfloor l/2 \rfloor + 1 - \nu}, t_{B_l, \lfloor l/2 \rfloor - \nu}), (\nu = 1, \dots, \lfloor l/2 \rfloor - 1).$$

Next, we note that the coefficients $lt - 2$ and $\frac{1}{2(l-1)}\{2(2l-1)t^2 - 5lt + 2l\}$ in (8.6) take positive values on the interval $(2/l, 1]$. Hence, by applying intermediate value theorem to the polynomial $f_{D_l}^+(t)$, we obtain the results. \square

Next, from Corollary 5.4, we have

$$t_{D_l, \lfloor l/2 \rfloor + 1 - \nu} \in (t_{B_{l-1}, \lfloor l/2 \rfloor + 1 - \nu}, t_{B_l, \lfloor l/2 \rfloor + 1 - \nu}), (\nu = 1, \dots, \lfloor l/2 \rfloor).$$

Therefore, we have the following inequalities

$$t_{D_l, \lfloor l/2 \rfloor} < t_{D_l, \lfloor l/2 \rfloor}^+ < \dots < t_{D_l, 2} < t_{D_l, 2}^+ < t_{D_l, 1} < 1.$$

\square

\square

9. APPENDIX I.

From [F-R1]§8, we make a list of f -polynomials of types A_l, B_l and D_l . Table A contains three infinite series A_l ($l \geq 1$), B_l ($l \geq 2$) and D_l ($l \geq 4$). Table B contains the remaining exceptional types E_6, E_7, E_8, F_4 and G_2 and non-crystallographic types H_3, H_4 and $I_2(p)$. We note that in [F-Z] Proposition 3.7 the f -polynomials of crystallographic types are substantially determined.

Table A

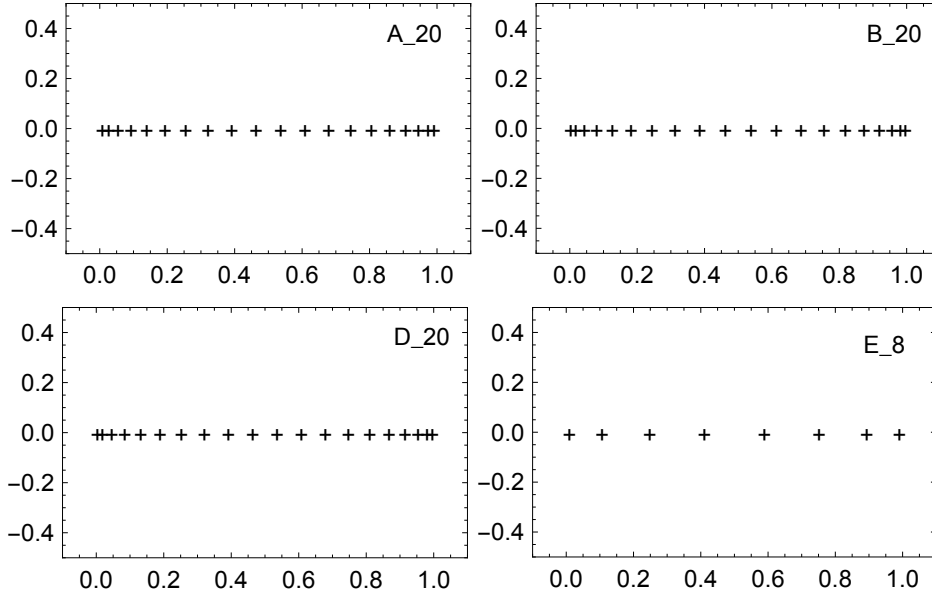
$$\begin{aligned} f_{A_l}(t) &= \sum_{k=0}^l (-1)^k \frac{1}{l+2} \binom{l}{k} \binom{l+k+2}{k+1} t^k, \\ f_{B_l}(t) &= \sum_{k=0}^l (-1)^k \binom{l}{k} \binom{l+k}{k} t^k, \\ f_{D_l}(t) &= \sum_{k=0}^l (-1)^k \left(\binom{l}{k} \binom{l+k-1}{k} + \binom{l-2}{k-2} \binom{l+k-2}{k} \right) t^k. \end{aligned}$$

Table B

$$\begin{aligned} f_{E_6}(t) &= 1 - 42t + 399t^2 - 1547t^3 + 2856t^4 - 2499t^5 + 833t^6, \\ f_{E_7}(t) &= 1 - 70t + 945t^2 - 5180t^3 + 14105t^4 - 20202t^5 + 14560t^6 - 4160t^7, \\ f_{E_8}(t) &= 1 - 128t + 2408t^2 - 17936t^3 + 67488t^4 - 140448t^5 + 163856t^6 - 100320t^7 + 25080t^8, \\ f_{F_4}(t) &= 1 - 28t + 133t^2 - 210t^3 + 105t^4, \\ f_{G_2}(t) &= 1 - 8t + 8t^2, \\ f_{H_3}(t) &= 1 - 18t + 48t^2 - 32t^3, \\ f_{H_4}(t) &= 1 - 64t + 344t^2 - 560t^3 + 280t^4, \\ f_{I_2(p)}(t) &= 1 - (p+2)t + (p+2)t^2. \end{aligned}$$

10. APPENDIX II

The zero loci in the complex plane of the f -polynomial $f_P(t)$ for types A_{20}, B_{20}, D_{20} and E_8 are exhibited in the following figures, where the zeros are indicated by +.



11. ADDENDUM

In Remark 4.4 in [I-S], for an Artin monoid G_P^+ and a dual Artin monoid G_P^{dual+} of type P , the authors observed that the derivative at $t = 1$ of the skew-growth function $N_{G_P^+}(t)$ for the Artin monoid G_P^+ coincides with that of the dual Artin monoid G_P^{dual+} . In addition to this observation, we have found the following equalities

$$N'_{G_P^+}(1) = N'_{G_P^{dual+}}(1) = \frac{(-1)^l}{|W|} (lh) \prod_{i=2}^l (e_i - 1),$$

where h is the Coxeter number and e_1, e_2, \dots, e_l are the exponents of the corresponding finite reflection group W of type P .

12. ADDENDUM

In [I-S], we have studied the polynomial $\hat{N}_{G_{D_l}^+}(t)$ ($= f_{D_l}^+(t)/(1-t)$). We note that the polynomial $\hat{N}_{G_{D_l}^+}(t)$ satisfies the following Fuchsian ordinary differential equation of third-order. The proof is left to the reader.

$$\begin{aligned} & t(t-1)\{2(l-1)t-l\} \frac{d^3 y}{dt^3} + \{(l+8)(l-1)t^2 - (l^2 + 6l - 2)t + 2l\} \frac{d^2 y}{dt^2} \\ & - \{(l-1)(2l^2 - 5l - 2)t - (l^3 - 2l^2 - l - 2)\} \frac{dy}{dt} - (l-1)^3(l+2)y = 0. \end{aligned}$$

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